ON THE USE OF LATENT VECTORS IN THE ANALYSIS OF GROUP DIVISIBLE AND L_2 DESIGNS

BY DAMARAJU RAGHAVARAO*

University of Bombay

1. INTRODUCTION

AN incomplete block design is said to be partially balanced (P.B.I.B.) if it satisfies the following conditions [*cf.* Bose and Shimamoto (1952)]:

(i) The experimental material is divided into b blocks of k plots each, different treatments being applied to the units in the same block.

(ii) There are v treatments each of which occurs in r blocks.

(iii) There can be established relations of association between any two treatments satisfying the following requirements;

(*a*) Two treatments are either 1st, 2nd, ..., or m-th associates.

(*b*) Each treatment has exactly n_i, i-th associates.

(*c*) Given any two treatments which are i-th associates, the number of treatments common to the j-th associates of the first and the k-th associates of the second is p_{jk}^i and is independent of the pair of treatments with which we start.

Also, $p_{jk}^i = p_{kj}^i$

(iv) Two treatments which are i-th associates occur together in exactly λ_i blocks.

P.B.I.B. designs were first introduced into the experimental designs by Bose and Nair (1939). Bose and Nair have shown that

$$vr = bk, \sum_{i=1}^{m} n_i = v - 1, \sum_{i=1}^{m} n_i\lambda_i = r(k-1),$$

$$\sum_{k=1}^{m} p_{jk}^i = n_j - \delta_{ij}, n_i p_{jk}^i = n_j p_{ki}^j = n_k p_{ij}^k, \quad (1.1)$$

$$i, j, k = 1, 2, \ldots, m,$$

* This work was financially supported by a Government of India Research Fellowship.

where δ_{ij} is the Kronecker delta taking the value 1 or 0 according as $i=j$ or $i \neq j$ respectively. P.B.I.B. designs with two associate classes have been classified by Bose and Shimamoto (1952) into five categories, *viz.*, (1) Group divisible, (2) Simple, (3) Triangular, (4) Latin square (L_i), and (5) Cyclic. Tables of P.B.I.B. designs with two associate classes have been prepared by Bose, Clatworthy and Shrikhande (1954).

While considering the analysis of P.B.I.B. designs with two associate classes, Bose and Shimamoto (1952) have introduced two new parameters c_1 and c_2 to facilitate the analysis. The usual method of finding the solution of the normal equation is of somewhat tedious nature and the author has obtained a simple method of solving the normal equations for group divisible and L_2 designs based on the latent roots and vectors of the C matrix. This method of solving the normal equations can be utilized for a wider class of designs and results in those directions will be communicated in subsequent papers.

2. Intrablock Model and Preliminaries

Let $N=(n_{ij})$ be the incidence matrix of the design where $n_{ij}=1$ or 0 according as the i-th treatment occurs in the j-th block or not. For the intrablock analysis, we assume the model

$$y_{ij} = m + b_j + t_i + e_{ij}, \tag{2.1}$$

where y_{ij} is the yield of the plot of the j-th block to which the i-th treatment is applied, m is the general effect, b_j is the effect of the j-th block, t_i is the effect of the i-th treatment and e_{ij}'s are independent normal variates with mean zero and variance σ^2. Let T_i be the total yield of all the plots having the i-th treatment, B_j be the total yield of all the plots of the j-th block and $\hat{t}_i$ be an estimate of t_i. Further denote the column vectors $\{T_1, T_2, \ldots, T_v\}$, $\{B_1, B_2, \ldots, B_b\}$, $\{t_1, t_2, \ldots, t_v\}$ and $\{\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_v\}$ by $\boldsymbol{T}$, $\boldsymbol{B}$, $\boldsymbol{t}$ and $\hat{\boldsymbol{t}}$ respectively. It is well known that the reduced normal equations for the intrablock estimates of the treatment contrasts are

$$\boldsymbol{Q} = \boldsymbol{C}\hat{\boldsymbol{t}}, \tag{2.2}$$

where

$$\boldsymbol{Q} = \boldsymbol{T} - k^{-1}\boldsymbol{NB}, \tag{2.3}$$

and

$$\boldsymbol{C} = r\boldsymbol{I}_v - k^{-1}\boldsymbol{NN}', \tag{2.4}$$

where $\boldsymbol{I}_v$ is the identity matrix of order v and N' is the transpose of N. Shah (1959) has proved that

$$\hat{t} = (C + aE_{vv})^{-1} Q, \tag{2.5}$$

where E_{mn} is an $m \times n$ matrix with positive unit elements everywhere and a is any non-zero real number, is a solution of (2.2).

Let $\theta_0 = rk, \theta_1, \theta_2, \ldots, \theta_m$ be the distinct latent roots of NN' with multiplicities $\alpha_0 = 1, \alpha_1, \alpha_2, \ldots, \alpha_m$ respectively. Then it follows that $\phi_0 = 0, \phi_1 = r - \theta_1/k, \phi_2 = r - \theta_2/k, \ldots, \phi_m = r - \theta_m/k$ are the latent roots of C with multiplicities $\alpha_0 = 1, \alpha_1, \alpha_2, \ldots, \alpha_m$ respectively. It is well known that NN' can be written as

$$NN' = \theta_0 A_0 + \theta_1 A_1 + \theta_2 A_2 + \ldots + \theta_m A_m, \tag{2.6}$$

where A_i's $(i = 0, 1, 2, \ldots, m)$ are idempotent matrices, such that

$$A_i' A_j = \mathbf{0}_{vv}, \quad (i \neq j = 0, 1, 2, \ldots, m) \tag{2.7}$$

where $\mathbf{0}_{mn}$ is an $m \times n$ null matrix, and

$$A_0 + A_1 + A_2 + \ldots + A_m = I_v. \tag{2.8}$$

In fact, if $x_1, x_2, \ldots, x_{\alpha_i}$ is a set of normalized orthogonal latent vectors corresponding to the root θ_i of NN', then

$$\sum_{j=1}^{\alpha_i} x_j x_j' = A_i. \tag{2.9}$$

It is obvious that $A_0 = E_{vv}/v$.

From the structure of C given by (2.4), we have

$$C = \phi_1 A_1 + \phi_2 A_2 + \ldots + \phi_m A_m. \tag{2.10}$$

The solution $\hat{t}$ of (2.5) now becomes

$$\hat{t} = \{(av)^{-1} A_0 + \phi_1^{-1} A_1 + \phi_2^{-1} A_2 + \ldots + \phi_m^{-1} A_m\} Q \tag{2.11}$$

Thus, to find the estimates of treatment effects, we must find the matrices A_i's $(i = 1, 2, \ldots, m)$ for the various designs. In the next two sections we derive A_i's for group divisible and L_2 designs.

3. Analysis of Group Divisible Designs

For a group divisible design $v = mn$ and the treatments can be divided into m groups of n treatments each, such that any two treatments of the same group are first associates while two treatments from different groups are second associates [*cf.* Bose and Connor (1952)]. It was shown by Connor and Clatworthy (1954) that for group divisible designs

$$\theta_1 = rk - v\lambda_2, \qquad \theta_2 = r - \lambda_1,$$
$$\alpha_1 = m - 1, \qquad \alpha_2 = m(n-1). \tag{3.1}$$

The treatments of the group divisible design can be so arranged that

$$NN' = I_m X (A - B) + E_{mm} X B, \tag{3.2}$$

where 'X' denotes the direct (or Kronecker) product of matrices,

$$A = (r - \lambda_1) I_n + \lambda_1 E_{nn}, \text{ and}$$
$$B = \lambda_2 E_{nn} \tag{3.3}$$

A moments consideration at NN' shows that the $(m-1)$ column 1 vectors of

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} & \cdots & \frac{1}{\sqrt{m(m-1)}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} & \cdots & \frac{1}{\sqrt{m(m-1)}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{12}} & \cdots & \frac{1}{\sqrt{m(m-1)}} \\ 0 & 0 & -\frac{3}{\sqrt{12}} & \cdots & \frac{1}{\sqrt{m(m-1)}} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & -\frac{(m-1)}{\sqrt{m(m-1)}} \end{bmatrix} X \frac{E_{n1}}{\sqrt{n}} \tag{3.4}$$

form a complete set of $(m-1)$ normalized orthogonal latent vectors of NN' corresponding to the root θ_1. Hence

$$A_1 = UU' = I_m X n^{-1} E_{nn} - v^{-1} E_{vv} \tag{3.5}$$

and

$$A_2 = I_v - A_0 - A_1$$
$$= I_m X \{I_n - n^{-1} E_{nn}\} \tag{3.6}$$

From (2.11) we get

$$\hat{t} = \left[(av^2)^{-1} E_{vv} + I_m \; X \left\{ \phi_2^{-1} I_n + \frac{\phi_2 - \phi_1}{\phi_1 \phi_2 n} E_{nn} \right\} - v\phi_1)^{-1} E_{vv} \right] Q \tag{3.7}$$

Since a can be arbitrarily chosen, let us put $av = \phi_1$. Then (3.7) reduces to

$$\hat{t} = \left[I_m \; X \left\{ \phi_2^{-1} I_n + \frac{\phi_2 - \phi_1}{\phi_1 \phi_2 n} E_{nn} \right\} \right] Q. \tag{3.8}$$

If G_i is the sum of all Q's of all treatments belonging to the same group as the treatment i, from equation (3.8) we have

$$\hat{t}_i = \frac{Q_i}{\phi_2} + \frac{\phi_2 - \phi_1}{\phi_1 \phi_2 n} G_i. \tag{3.9}$$

The further analysis can be completed in the usual way. From (3.8) it follows that

$$\text{Var}\,(\hat{t}_i - \hat{t}_j) = \frac{2\sigma^2}{\phi_2}, \quad \text{if treatments } i \text{ and } j \text{ are first associates;}$$

$$= 2 \left\{ \frac{1}{\phi_2} + \frac{\phi_2 - \phi_1}{\phi_1 \phi_2 n} \right\} \sigma^2, \quad \text{otherwise.}$$

It is interesting to observe that the variances of treatment comparisons depend only on ϕ_1, ϕ_2 and n.

4. Analysis of L_2 Designs

A L_2 design has $v = s^2$ treatments arranged in an $s \times s$ square such that treatments in the same row and the same column are the first associates and the others are the second associates [*cf.* Bose and Shimamoto (1952)].

Connor and Clatworthy (1954) have proved that for these designs

$$\theta_1 = (r - \lambda_1) + (s - 1)(\lambda_1 - \lambda_2), \quad \theta_2 = r - 2\lambda_1 + \lambda_2$$

$$\alpha_1 = 2(s - 1), \quad \alpha_2 = (s - 1)^2. \tag{4.1}$$

The treatments of the L_2 designs can also be so arranged that

$$NN' = I_s \, X (A - B) + E_{ss} \, X \, B, \tag{4.2}$$

where

$$A = (r - \lambda_1) I_s + \lambda_1 E_{ss}, \quad \text{and}$$

$$B = (\lambda_1 - \lambda_2) I_s + \lambda_2 E_{ss}. \tag{4.3}$$

In this case we see that the $(s-1)$ column vectors of

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} & \cdots & \frac{1}{\sqrt{s(s-1)}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} & \cdots & \frac{1}{\sqrt{s(s-1)}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{12}} & \cdots & \frac{1}{\sqrt{s(s-1)}} \\ 0 & 0 & -\frac{3}{\sqrt{12}} & \cdots & \frac{1}{\sqrt{s(s-1)}} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & -\frac{(s-1)}{\sqrt{s(s-1)}} \end{bmatrix} X \frac{E_{s1}}{\sqrt{s}} \tag{4.4}$$

and the $(s-1)$ column vectors of

$$U^* = \frac{E_{s1}}{\sqrt{s}} X \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} & \cdots & \frac{1}{\sqrt{s(s-1)}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} & \cdots & \frac{1}{\sqrt{s(s-1)}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{12}} & \cdots & \frac{1}{\sqrt{s(s-1)}} \\ 0 & 0 & -\frac{3}{\sqrt{12}} & \cdots & \frac{1}{\sqrt{s(s-1)}} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & -\frac{(s-1)}{\sqrt{s(s-1)}} \end{bmatrix}$$

orm the complete set of $2(s-1)$ normalized orthogonal latent vectors of NN' corresponding to the root θ_1. Hence

$$\begin{aligned} A_1 &= UU' + U^*U^{*\prime} \\ &= I_s X s^{-1}E_{ss} + s^{-1}E_{ss} X I_s - 2v^{-1}E_{vv}, \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} A_2 &= I_v - A_0 - A_1 \\ &= I_s X I_s - I_s X s^{-1}E_{ss} - s^{-1}E_{ss} X I_s + v^{-1}E_{vv} \end{aligned} \tag{4.7}$$

From (2.11) and suitably choosing a we get

$$\hat{t} = \left[\phi_2^{-1} I_v + I_s X \frac{\phi_2 - \phi_1}{\phi_1\phi_2 s} E_{ss} + \frac{\phi_2 - \phi_1}{\phi_1\phi_2 s} E_{ss} \; X I_s\right] Q \qquad (4.8)$$

If G_i is the sum of Q's of all treatments occurring in the same row as i and G_i' is the sum of Q's of all treatments occurring in the same column as i, from equation (4.8) we get

$$\hat{t}_i = \frac{Q_i}{\phi_2} + \frac{\phi_2 - \phi_1}{\phi_1\phi_2 s}(G_i + G_i') \qquad (4.9)$$

The further analysis can be completed in the usual way. From (4.8), it follows that

$$\text{Var}(\hat{t}_i - \hat{t}_j) = 2\left\{\frac{1}{\phi_2} + \frac{\phi_2 - \phi_1}{\phi_1\phi_2 s}\right\}\sigma^2, \quad \text{if the treatments } i \text{ and } j \text{ are first associates;}$$

$$= 2\left\{\frac{1}{\phi_2} + \frac{2(\phi_2 - \phi_1)}{\phi_1\phi_2 s}\right\}\sigma^2, \quad \text{otherwise.}$$

It is interesting to observe that the variances of treatment comparisons depend only on ϕ_1, ϕ_2 and s.

I am thankful to Professor M. C. Chakrabarti for his kind interest in this work.

5. Summary

In this paper the analysis of group divisible and L_2 designs is derived with the help of the latent roots and vectors of the usual C matrix in an elegant way.

6. References

1. Bose, R. C. and Connor, W. S. "Combinatorial properties of group divisible incomplete block designs," *Ann. Math. Stat.*, 1952, **23**, 367–83.
2. —— and Nair, K. R. "Partially balanced incomplete block designs," *Sankhya*, 1939, **4**, 337–72.
3. —— and Shimamoto .. "Classification and analysis of partially balanced incomplete block designs with two associate classes," *J.A.S.A.*, 1952, **47**, 151–84.
4. ——, Clatworthy, W. H. and Shrikhande, S. S. *Tables of Partially Balanced Incomplete Block Designs*, 1954.
5. Connor, W. S. and Clatworthy, W. H. "Some theorems for partially balanced designs," *Ann. Math. Stat.*, 1954, **26**, 100–12.
6. Shah, B. V. .. "A generalization of partially balanced incomplete block designs," *Ibid.*, 1959, **30**, 1041–50.

A NOTE ON FRACTIONAL REPLICATE OF BALANCED INCOMPLETE BLOCK DESIGNS

By A. S. Chopra and M. N. Das

I.A.R.S., New Delhi

The B.I.B. designs, introduced by Yates (1936), have been extensively used in experimental investigations in the biological and other sciences. In certain situations, however, this design requires a larger number of replications than can be conveniently adopted in experimental investigations, owing to the limited resources at the disposal of the experimenter. In such situations, it becomes necessary to develop other types of incomplete block designs which require lesser number of replications than those for a B.I.B. design. Accordingly, incomplete block designs like the lattice design, P.B.I.B. design, rectangular lattices, etc., have been developed which provide suitable designs requiring a small number of replications and are, consequently, suitable for adoption in many experimental situations. In the present note, an attempt has been made to derive a class of P.B.I.B. designs from B.I.B. designs developable from two or more initial blocks by taking one or more, but not all of the initial blocks of the B.I.B. designs. The use of this device thus provides P.B.I.B. designs in which the number of replications is only a fraction of that required for the parent B.I.B. designs.

Fractions of all such B.I.B. designs presented by Fisher and Yates (1952), Kitagawa and Michiwo (1955) and by exploring the series given by Bose (1942) have been considered in the present note. When there is no problem regarding the construction of such designs, their analysis presents considerable difficulty. Though all the designs are P.B.I.B. designs, the difficulty of analysis is due to the fact that some of the λ's are equal and hence finding out the associates for different treatments is not straightforward. Though the blocks chosen for the present investigations are obtainable from only one initial block, other designs derivable from two or more initial blocks are also possible. In these designs, both the number of associate classes and the association scheme remain the same as in the case of the design obtainable from only one initial block.

The intra-block analysis of each such designs has been worked out by providing solutions of normal equations and giving expressions for the variance of treatment differences. The solution together with the parameters of the designs have been presented in Tables I–III.

(*a*) In Table II $i + (x, y)$ means $(i + x, i + y)$ that is, each of the numbers in the bracket is increased by i and reduced mod (v).

(*b*) Q_i is the adjusted total of the i-th treatment, $\Sigma_p Q_i$ is the sum of Q's of those treatments which are the p-th associates of the i-th treatment.

(*c*) The different variances, Var $(t_i - t_j)$, are obtained by collecting the coefficients of Q_i and Q_j in the estimate of $t_i - t_j$ and subtracting. For example, for Design No. 1,

$$\text{Var}\,(t_i - t_j) = \frac{2\,(14 - a_j)}{45}\,\sigma^2,$$

where $a_j = 1$, or 0 accordingly as the treatment t_j is the 1st associate or 2nd associate of the i-th treatment.

(*d*) Efficiency factor (E.F.) of the designs has been obtained from the mean variance of the treatment differences as usual.

ACKNOWLEDGEMENT

We are grateful to Dr. G. R. Seth, Statistical Adviser, I.C.A.R., for suggesting the problem and providing us an opportunity to conduct the research.

REFERENCES

1. Bose, R. C. .. "On some new series of balanced incomplete block designs," *Bull. Cal. Math. Soc.*, 1942, **34**.

2. Fisher, R. A. and Yates, F. *Statistical Tables for Biological, Agricultural and Medical Research*, Olive and Boyd, Edinburg, 1952.

3. Kitagawa, T. and Michiwo, M. *Tables for the Design of Factorial Experiments*, Dover Publications, Inc., New York, 1955.

4. Yates, F. .. "Incomplete randomised designs," *Ann. of Eug.*, 1936, **7**.

TABLE I

Parameters and initial blocks of the original balanced incomplete block designs which can be fractionated

Sl. No.	v	b	r	k	λ	Initial Blocks
1	9	18	8	4	3	$(1, 2x+1, 2, x+2)$ and $(x, 2x+2, 2x, x+1)$ by adding the polynomials $ax+b\ (a, b = 0, 1, 2)$ mod 3
2	9	18	8	4	3	(1, 2, 3, 5) and (1, 4, 6, 9)
3	9	18	10	5	5	(1, 2, 3, 4, 8) and (1, 2, 4, 6, 7)
4	13	26	6	3	1	(1, 3, 9) and (1, 2, 11)
5	13	39	12	4	3	(1, 5, 8, 12); (1, 2, 9, 10) and (1, 3, 4, 6)
6	16	48	15	5	4	$(1, x^3, x^3+x^2+x+1, x^2+1, x+1)$; $(x, x^3+1, x^2+x+1, x^3+x, x^2+x)$ and $(x^2, x^3+x+1, x^3+x^2+x, x^3+x^2+1, x^3+x^2)$ by adding the plynomials $ax^3+bx^2+cx+d\ (a, b, c, d = 0, 1)$ mod 2
7	16	80	15	3	2	(1, 2, 10); (1, 2, 14); (1, 3, 11); (1, 4, 11) and (1, 4, 15)
8	17	68	16	4	3	(1, 4, 13, 16); (1, 2, 10, 12); (1, 7, 8, 14) and (1, 2, 5, 6)
9	19	57	9	3	1	(1, 7, 11); (1, 2, 13) and (1, 3, 6)
10	19	57	12	4	2	(1, 2, 8, 12); (1, 3, 4, 15) and (1, 5, 7, 10)
11	25	100	16	4	2	$(0, 1, 4x+1, x+3)$; $(0, x, 3x+3, x+2)$; $(0, 3x+2, 2x+1, 2)$ and $(0, x+1, 2x+4, 2x)$ by adding the polynomials $ax+b\ (a, b=0, 1, 2, 3, 4)$ mod 5
11*a*	25	50	8	4	1	$(0, 1, x, 4x+4)$ and $(0, 3, 2x, 3x+3)$ by adding the polynomials $ax+b\ (a, b = 0, 1, 2, 3, 4)$ mod 5
12	31	155	20	4	2	(1, 2, 6, 26); (1, 4, 14, 16); (1, 9, 10, 15); (1, 12, 25, 28) and (1, 3, 11, 20)
13	31	93	15	5	2	(1, 2, 4, 8, 16); (1, 4, 10, 15, 22) and (1, 5, 6, 14, 16)
14	41	82	10	5	1	(1, 10, 16, 18, 37) and (1, 9, 31, 32, 35)
15	41	164	20	5	2	(1, 10, 16, 18, 37); (1, 5, 11, 27, 28); (1, 4, 5, 17, 35) and (1, 8, 10, 13, 21)

TABLE II

Parameters of the designs taken as a fraction of a balanced incomplete block design

Design No.	$v=b$	$k=r$	Initial Block	No. of associate classes	λ_i	n_i	Association scheme Treatment numbers forming different associations of the i-th treatment
1	9	4	$1, 2x+1, 2, x+2$	2	2, 1	4, 4	$i+(x, x+1, 2x, 2x+2)$ and $i+(1, 2, x+2, 2x+1)$*
2	9	4	1, 2, 3, 5	4	2, 2, 1, 1	2, 2, 2, 2	$i+(1, 8)$; $i+(2, 7)$, $i+(3, 6)$ and $i+(4, 5)$
3	9	5	1, 2, 3, 4, 8	4	3, 3, 2, 2	2, 2, 2, 2	$i+(1, 8)$; $i+(2, 7)$, $i+(3, 6)$ and $i+(4, 5)$
4	13	3	1, 2, 11	2	1, 0	6, 6	$i+(2, 5, 6, 7, 8, 11)$ and $i+(1, 3, 4, 9, 10, 12)$
5	13	4	1, 5, 8, 12	3	2, 1, 0	4, 4, 4	$i+(4, 6, 7, 9)$; $i+(2, 3, 10, 11)$ and $i+(1,5,8,12)$
6	16	5	$1, x^3, x^3+x^2+x+1, x^2+1, x+1$	2	2, 0	10, 5,	$i+(x, x^2, x^2+x, x^2+x+1, x^3+1, x^3+x, x^3+x+1, x^3+x^2, x^3+x^2+1, x^3+x^2+x)$; $i+(1, x+1, x^2+1, x^3, x^3+x^2+1)$ †
7	16	3	1, 2, 10	8	1, 1, 1, 0, 0 0, 0, 0	2, 2, 2, 2 2,2,2,1	$i+(1, 15)$; $i+(2, 14)$; $i+(3, 13)$; $i+(4, 12)$; $i+(5, 11)$; $i+(6, 9)$ $i+(7, 9)$; $i+(8)$
8	17	4	1, 4, 13, 16	4	2, 1, 0, 0	4, 4, 4, 4	$i+(3, 5, 12, 14)$; $i+(2, 8, 9, 15)$; $i+(1, 4, 13, 16)$ and $i+(6, 7, 10, 11)$
9	19	3	1, 7, 11	3	1, 0, 0	6, 6, 6	$i+(4, 6, 9, 10, 13, 15)$; $i+(2, 3, 5, 14, 16, 17)$ and $i+(1, 7, 8, 11, 12, 18)$

10	19	4	1, 2, 8, 12	3	1, 1, 0	6, 6, 6, 6	$i+$ (1, 7, 8, 11, 12, 18); $i+$ (4, 6, 9, 10, 13, 15) and $i+$ (2, 3, 5, 14, 16, 17)
11	25	4	0, 1, $4x+1$, $x+3$	4	1, 1, 0, 0	6, 6, 6, 6	$i+$ (1, 4, $x+3$, $x+4$, $4x+1$, $4x+2$); $i+$ (x, $x+2$, $2x+2$, $3x+3$, $4x$, $4x+3$); $i+$ (2, 3, $2x+1$, $2x+3$, $3x+2$, $3x+4$) and $i+$ ($x+1$, $2x$, $2x+4$, $3x$, $3x+1$, $4x+4$)‡
11 *a*	25	4	0, 1, x, $4x+4$	4	1, 1, 0, 0	6, 6, 6, 6	$i+$ (1, 4, $x+3$, $x+4$, $4x+1$, $4x+2$); $i+$ (x, $x+2$, $2x+2$, $3x+3$, $4x$, $4x+3$); $i+$ (2, 3, $2x+1$, $2x+3$, $3x+2$, $3x+4$) and $i+$ ($x+1$, $2x$, $2x+4$, $3x$, $3x+1$, $4x+4$)‡
12	31	4	1, 2, 6, 26	5	1, 1, 0, 0, 0	6, 6, 6, 6, 6	$i+$ (1, 5, 6, 25, 26, 30); $i+$ (4, 7, 11, 20, 24, 27); $i+$ (2, 10, 12, 19, 21, 29); $i+$ (3, 13, 15, 16, 18, 28) and $i+$ (8, 9, 14, 17, 22, 23)
13	31	5	1, 2, 4, 8, 16	3	1, 1, 0	10, 10, 10	$i+$ (1, 2, 4, 8, 15, 16, 23, 27, 29, 30); $i+$ (3, 6, 7, 12, 14, 17, 19, 24, 25, 28) and $i+$ (5, 9, 10, 11, 13, 18, 20, 21, 22, 26)
14 and 15	41	5	1, 10, 16, 18, 37	4	1, 1, 0, 0	10, 10, 10, 10	$i+$ (6, 14, 15, 17, 19, 22, 24, 26, 27, 35); $i+$ (2, 5, 8, 9, 20, 21, 32, 35, 36, 39); $i+$ (3, 7, 11, 12, 13, 28, 29, 30, 34, 38) and $i+$ (1, 4, 10, 16, 18, 23, 25, 31, 37, 40)

* 0, 1, 2, x, $x+1$, $x+2$, $2x$, $2x+1$, $2x+2$, mod (3) are nine treatments and $i=0, 1, \ldots\ldots, 2x+2$.

† 0, 1, x, $x+1$, x^2, x^2+1, x^2+x, x^2+x+1, x^3, x^3+x+1, x^3+x^2, x^3+x^2+x, x^3+x^2+x+1, mod (2) are 16 treatments, $i=0, 1, \cdots, x^3+x^2+x+1$.

‡ 0, 1, 2, 3, 4, x, $x+1$, $x+2$, $x+3$, $x+4$, $2x$, $2x+1$, $2x+2$, $2x+3$, $2x+4$, $3x$, $3x+1$, $3x+2$, $3x+3$, $3x+4$, $4x$, $4x+1$, $4x+2$, $4x+3$, $4x+4$ mod (5) are 25 treatments and $i=0, 1, \cdots, 4x+4$.

A design having the same number in the next table as in this table refers to the same design.

TABLE III

Solution and different variances

Design No.	Solution for t_i		Different variances	E.F.	E.F. of the B.I.B. design
1	$45t_i$	$=14\,Q_i + \Sigma_1 Q_i$	$2\,(14-a_j)\,\sigma^2/45$	0·833	0·84
2	$32355t_i$	$= 10136\,Q_i + 840\,\Sigma_1 Q_i + 780\,\Sigma_2 Q_i + 56\,\Sigma_4 Q_i$	$2\,(10136-a_j)\,\sigma^2/32355$	0·832	0·84
3	$253944t_i$	$= 5\,(11615\,Q_i + 552\,\Sigma_1 Q_i + 528\,\Sigma_2 Q_i + 23\,\Sigma_3\,Q_i)$	$2\,(58075-a_j)\,\sigma^2/253944$	0·896	0·90
4	$13t_i$	$= 7\,Q_i + \Sigma_1 Q_i$	$2\,(7-a_j)\,\sigma^2/13$	0·667	0·72
5	$2015t_i$	$= 4\,(180Q_i + 29\,\Sigma_1 Q_i + 17\,\Sigma_2\,Q_i)$	$2\,(720-a_j)\,\sigma^2/2015$	0·765	0·81
6	$192t_i$	$= 5\,(10Q_i + \Sigma_1\,Q_i)$	$2\,(50-a_j)\,\sigma^2/192$	0·808	0·85
7	$105664t_i$	$= 62404\,Q_i + 10803\,\Sigma_1 Q_i + 2106\,\Sigma_2 Q_i + 10304\,\Sigma_3 Q_i + 10140\,\Sigma_4 Q_i + 195\,\Sigma_5 Q_i - 1638\,\Sigma_6 Q_i - 429\Sigma_7 Q_i$	$2\,(62404-a_j)\,\sigma^2/105664$	0·632	0·71
8	$22321t_i$	$= 4\,(2014Q_i + 322\,\Sigma_1 Q_i + 146\,\Sigma_2 Q_i - 79\,\Sigma_3 Q_i)$	$2\,(8056-a_j)\,\sigma^2/22321$	0·728	0·80
9	$209t_i$	$= 3\,(40Q_i + 7\Sigma_1\,Q_i - \Sigma_2\,Q_i)$	$2\,(120-a_j)\,\sigma^2/209$	0·611	0·70
10	$1957t_i$	$= 4\,(166Q_i + 12\,\Sigma_1 Q_i + 11\,\Sigma_2 Q_i)$	$2\,(664-a_j)\,\sigma^2/1957$	0·772	0·79
11 & 11*a*	$22475t_i$	$= 4\,(1963Q_i + 179\,\Sigma_1 Q_i + 151\,\Sigma_2 Q_i + 22\Sigma_3 Q_i)$	$2\,(7852-a_j)\,\sigma^2/22475$	0·749	0·78
12	$254851t_i$	$= 4\,(22833\,Q_i + 2491\,\Sigma_1 Q_i + 2070\,\Sigma_2 Q_i + 346\,\Sigma_3 Q_i + 138\,\Sigma_4 Q_i)$	$2\,(91332 - a_j)\,\sigma^2/254851$	0·729	0·83
13	$8618t_i$	$= 5\,(437\,Q_i + 21\,\Sigma_1 Q_i + 19\,\Sigma_2 Q_i)$	$2\,(2185-a_j)\,\sigma^2/8618$	0·814	0·83
14 & 15	$168059t_i$	$= 5\,(8589Q_i + 373\Sigma_1 Q_i + 409\,\Sigma_2 Q_i - 46\Sigma_3 Q_i)$	$2\,(42945-a_j)\,\sigma^2/168059$	0·800	0·82

Note.—The values of a_j in any design is equal to the coefficient of $\Sigma_j\,Q_i$ in the corresponding solution for t_i.